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EXISTENCE OF POSITIVE SOLUTIONS FOR A THREE-POINT INTEGRAL BOUNDARY VALUE PROBLEM

FAOUZI HADDOUCHI, SLIMANE BENAICHA

Abstract. In this paper, by using the Krasnosel'skii's fixed-point theorem, we study the existence of at least one or two positive solutions to the three-point integral boundary value problem

$$u''(t) + a(t)f(u(t)) = 0,\ 0 < t < T,$$

$$u(0) = \beta u(\eta),\ u(T) = \alpha \int_0^\eta u(s)ds,$$

where $0 < \eta < T$, $0 < \alpha < \frac{2T}{\eta^2}$, $0 \leq \beta < \frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$ are given constants.

1. Introduction

We are interested in the existence of positive solutions of the following three-point integral boundary value problem (BVP):

$$u''(t) + a(t)f(u(t)) = 0,\ t \in (0,T), \tag{1.1}$$

$$u(0) = \beta u(\eta),\ u(T) = \alpha \int_0^\eta u(s)ds, \tag{1.2}$$

where $0 < \eta < T$ and $0 < \alpha < \frac{2T}{\eta^2}$, $0 \leq \beta < \frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$, and

(B1) $f \in C([0,\infty),[0,\infty))$;
(B2) $a \in C([0,T],[0,\infty))$ and there exists $t_0 \in [\eta,T]$ such that $a(t_0) > 0$.

Set

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u},\ f_\infty = \lim_{u\to\infty} \frac{f(u)}{u}. \tag{1.3}$$

The study of the existence of solutions of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by II'in and Moiseev [6, 7]. Since then, by applying the Leray-Schauder continuation theorem, nonlinear alternative of Leray-Schauder, or coincidence degree theory, many authors studied more general nonlinear multi-point BVPs, for example, [1, 2, 3, 4, 10, 11, 12, 13], and references therein.

Tariboon and Sitthiwirattham [14] proved the existence of at least one positive solution on the condition that f is either superlinear or sublinear for the following BVP

$$u''(t) + a(t)f(u(t)) = 0,\ t \in (0,1), \tag{1.4}$$

$$u(0) = 0,\ u(1) = \alpha \int_0^\eta u(s)ds, \tag{1.5}$$

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where $0<\eta<1$ and $0<\alpha<\frac{2}{\eta^2}$, $f\in C([0,\infty),[0,\infty))$, $a\in C([0,1],[0,\infty))$ and there exists $t_0\in[\eta,1]$ such that $a(t_0)>0$. Very recently, Haddouchi and Benaicha [5], investigated the following three-point BVP

$$u''(t)+a(t)f(u(t))=0,\ t\in(0,T), \tag{1.6}$$

$$u(0)=\beta u(\eta),\ u(T)=\alpha\int_0^\eta u(s)ds, \tag{1.7}$$

where $0<\eta<T$ and $0<\alpha<\frac{2T}{\eta^2}$, $0\leq\beta<\frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$, $f\in C([0,\infty),[0,\infty))$, $a\in C([0,T],[0,\infty))$ and there exists $t_0\in[\eta,T]$ such that $a(t_0)>0$, and improved the results in [14].

In [5], the authors used the Krasnoselskii's theorem to prove the following result:

Theorem 1.1 (See [5]). *Assume* (B1) *and* (B2) *hold, and* $0<\alpha<\frac{2T}{\eta^2}$, $0\leq\beta<\frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$. *If either*

(D1) $f_0=0$ *and* $f_\infty=\infty$ *(f is superlinear), or*

(D2) $f_0=\infty$ *and* $f_\infty=0$ *(f is sublinear)*

then problem (1.6),(1.7) *has at least one positive solution.*

Liu [9] used the fixed-point index theorem to prove the existence of at least one or two positive solutions to the three-point boundary value problem BVP

$$u''(t)+a(t)f(u(t))=0,\ t\in(0,1), \tag{1.8}$$

$$u(0)=0,\ u(1)=\beta u(\eta), \tag{1.9}$$

where $0<\eta<1$ and $0<\beta<\frac{1}{\eta}$.

Motivated by the results of [9, 5] the aim of this paper is to establish some simple criterions for the existence of positive solutions of the BVP (1.1),(1.2), under $f_0=f_\infty=\infty$ or $f_0=f_\infty=0$. We also obtain some existence results for positive solutions of the BVP (1.1),(1.2) under $f_0,f_\infty\notin\{0,\infty\}$.

The key tool in our approach is the following Krasnosel'skii's fixed point theorem in a cone [8].

Theorem 1.2 ([8]). *Let E be a Banach space, and let $K\subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, and let*

$$A:K\cap(\overline{\Omega}_2\backslash\Omega_1)\longrightarrow K$$

be a completely continuous operator such that either

(i) $\|Au\|\leq\|u\|,\ u\in K\cap\partial\Omega_1$, *and* $\|Au\|\geq\|u\|,\ u\in K\cap\partial\Omega_2$; *or*

(ii) $\|Au\|\geq\|u\|,\ u\in K\cap\partial\Omega_1$, *and* $\|Au\|\leq\|u\|,\ u\in K\cap\partial\Omega_2$

hold. Then A has a fixed point in $K\cap(\overline{\Omega}_2\backslash\ \Omega_1)$.

2. Preliminaries

To prove the main existence results we will employ several straightforward lemmas.

Lemma 2.1 (See [5]). *Let $\beta\neq\frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$. Then for $y\in C([0,T],\mathbb{R})$, the problem*

$$u''(t)+y(t)=0,\ t\in(0,T), \tag{2.1}$$

$$u(0)=\beta u(\eta),\ u(T)=\alpha\int_0^\eta u(s)ds \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) &= \frac{\beta(2T-\alpha\eta^2)-2\beta(1-\alpha\eta)t}{(\alpha\eta^2-2T)-\beta(2\eta-\alpha\eta^2-2T)}\int_0^\eta(\eta-s)y(s)ds \\ &+\frac{\alpha\beta\eta-\alpha(\beta-1)t}{(\alpha\eta^2-2T)-\beta(2\eta-\alpha\eta^2-2T)}\int_0^\eta(\eta-s)^2y(s)ds \\ &+\frac{2(\beta-1)t-2\beta\eta}{(\alpha\eta^2-2T)-\beta(2\eta-\alpha\eta^2-2T)}\int_0^T(T-s)y(s)ds-\int_0^t(t-s)y(s)ds. \end{aligned}$$

Lemma 2.2 (See [5])**.** *Let $0<\alpha<\frac{2T}{\eta^2}$, $0\le\beta<\frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$. If $y\in C([0,T],[0,\infty))$, then the unique solution u of (2.1)-(2.2) satisfies $u(t)\ge 0$ for $t\in[0,T]$.*

Remark 2.3. *In view of Lemma 2.3 of [5], if $\alpha>\frac{2T}{\eta^2}$, $\beta\ge 0$ and $y\in C([0,T],[0,\infty))$, then (2.1)-(2.2) has no positive solution. Hence, in this paper, we assume that $\alpha\eta^2<2T$ and $0\le\beta<\frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$.*

Lemma 2.4 (See [5])**.** *Let $0<\alpha<\frac{2T}{\eta^2}$, $0\le\beta<\frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$. If $y\in C([0,T],[0,\infty))$, then the unique solution u of (2.1)-(2.2) satisfies*

$$\min_{t\in[\eta,T]}u(t)\ge\gamma\|u\|,\ \|u\|=\max_{t\in[0,T]}|u(t)|, \tag{2.3}$$

where

$$\gamma:=\min\left\{\frac{\eta}{T},\frac{\alpha(\beta+1)\eta^2}{2T},\frac{\alpha(\beta+1)\eta(T-\eta)}{2T-\alpha(\beta+1)\eta^2}\right\}\in(0,1). \tag{2.4}$$

In the rest of this article, we assume that $0<\alpha<\frac{2T}{\eta^2}$, $0\le\beta<\frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$. Let $E=C([0,T],\mathbb{R})$, and only the sup norm is used. It is easy to see that the BVP (1.1),(1.2) has a solution $u=u(t)$ if and only if u is a fixed point of operator A, where A is defined by

$$\begin{aligned} Au(t)=&\frac{\beta(2T-\alpha\eta^2)-2\beta(1-\alpha\eta)t}{(\alpha\eta^2-2T)-\beta(2\eta-\alpha\eta^2-2T)}\int_0^\eta(\eta-s)a(s)f(u(s))ds \\ &+\frac{\alpha\beta\eta-\alpha(\beta-1)t}{(\alpha\eta^2-2T)-\beta(2\eta-\alpha\eta^2-2T)}\int_0^\eta(\eta-s)^2a(s)f(u(s))ds \\ &+\frac{2(\beta-1)t-2\beta\eta}{(\alpha\eta^2-2T)-\beta(2\eta-\alpha\eta^2-2T)}\int_0^T(T-s)a(s)f(u(s))ds \\ &-\int_0^t(t-s)a(s)f(u(s))ds. \end{aligned} \tag{2.5}$$

Denote

$$K=\left\{u\in E: u\ge 0,\min_{t\in[\eta,T]}u(t)\ge\gamma\|u\|\right\}, \tag{2.6}$$

where γ is defined in (2.4). It is obvious that K is a cone in E. Moreover, by Lemma 2.2 and Lemma 2.4, $AK\subset K$. It is also easy to check that $A:K\to K$ is completely continuous.

In what follows, for the sake of convenience, set

$$\Lambda_1=\frac{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}{[2(\beta+1)+T^{-1}\beta\eta(\alpha\eta+2)+\alpha\beta T]\int_0^T T(T-s)a(s)ds},$$

$$\Lambda_2 = \frac{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}{2\gamma\eta\int_\eta^T (T-s)a(s)ds}.$$

3. The existence results of the BVP (1.1),(1.2) for the case: $f_0 = f_\infty = \infty$ or $f_0 = f_\infty = 0$

Now we establish conditions for the existence of positive solutions for the BVP (1.1),(1.2) under $f_0 = f_\infty = \infty$ or $f_0 = f_\infty = 0$.

Theorem 3.1. *Assume that the following assumptions are satisfied.*

- (H1) $f_0 = f_\infty = \infty$.
- (H2) *There exist constants* $\rho_1 > 0$ *and* $M_1 \in (0, \Lambda_1]$ *such that* $f(u) \le M_1\rho_1$, *for* $u \in [0, \rho_1]$.

Then, the problem (1.1)-(1.2) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < \rho_1 < \|u_2\|.$$

Proof. Since, $f_0 = \infty$, then for any $M_\star \in [\Lambda_2, \infty)$, there exists $\rho_\star \in (0, \rho_1)$ such that $f(u) \ge M_\star u$, $0 < u \le \rho_\star$.

Set $\Omega_{\rho_\star} = \{u \in E : \|u\| < \rho_\star\}$. By (2.5) and in view of the proof of Theorem 3.1 in [5] , for any $u \in K \cap \partial\Omega_{\rho_\star}$, we obtain

$$\begin{aligned}
Au(\eta) &= \frac{2\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^T (T-s)a(s)f(u(s))ds \\
&\quad -\frac{\alpha\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^\eta (\eta^2-2\eta s+s^2)a(s)f(u(s))ds \\
&\quad -\frac{2T-\alpha\eta^2}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^\eta (\eta-s)a(s)f(u(s))ds \\
&= \frac{2\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_\eta^T (T-s)a(s)f(u(s))ds \\
&\quad +\frac{2(T-\eta)}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^\eta sa(s)f(u(s))ds \\
&\quad +\frac{\alpha\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^\eta s(\eta-s)a(s)f(u(s))ds \\
&\ge \frac{2\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_\eta^T (T-s)a(s)f(u(s))ds \\
&\ge \rho_\star\gamma M_\star\frac{2\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_\eta^T (T-s)a(s)ds \\
&= \rho_\star M_\star \Lambda_2^{-1} \\
&\ge \rho_\star = \|u\|.
\end{aligned}$$

Thus

$$\|Au\| \ge \|u\|, \quad \text{for } u \in K \cap \partial\Omega_{\rho_\star}. \tag{3.1}$$

Now, since $f_\infty = \infty$, then for any $M^\star \in [\Lambda_2, \infty)$, there exists $\rho^\star > \rho_1$ such that $f(u) \ge M^\star u$, for $u \ge \gamma\rho^\star$.

Set $\Omega_{\rho^\star} = \{u \in E : \|u\| < \rho^\star\}$. Then, for any $u \in K \cap \partial\Omega_{\rho^\star}$, we have

$$\begin{aligned} Au(\eta) &\geq \frac{2\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_\eta^T (T-s)a(s)f(u(s))ds \\ &\geq \rho^\star\gamma M^\star \frac{2\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_\eta^T (T-s)a(s)ds \\ &= \rho^\star M^\star \Lambda_2^{-1} \\ &\geq \rho^\star = \|u\|. \end{aligned}$$

Which implies

$$\|Au\| \geq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_{\rho^\star}. \tag{3.2}$$

Finally, set $\Omega_{\rho_1} = \{u \in E : \|u\| < \rho_1\}$. From (H2), (2.5) and the proof of Theorem 3.1 in [5], for any $u \in K \cap \partial\Omega_{\rho_1}$, we have

$$\begin{aligned} Au(t) &\leq \frac{2\beta T + \alpha\beta\eta^2}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^\eta (\eta-s)a(s)f(u(s))ds \\ &\quad + \frac{\alpha\beta T}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^\eta (\eta-s)^2 a(s)f(u(s))ds \\ &\quad + \frac{2\beta\eta + 2T}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^T (T-s)a(s)f(u(s))ds \\ &\leq \frac{2T(\beta+1)+\beta\eta(\alpha\eta+2)}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^T (T-s)a(s)f(u(s))ds \\ &\quad + \frac{\alpha\beta T}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^T T(T-s)a(s)f(u(s))ds \\ &= \frac{2(\beta+1)+T^{-1}\beta\eta(\alpha\eta+2)+\alpha\beta T}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^T T(T-s)a(s)f(u(s))ds \\ &\leq M_1\rho_1 \frac{2(\beta+1)+T^{-1}\beta\eta(\alpha\eta+2)+\alpha\beta T}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_0^T T(T-s)a(s)ds \\ &= \rho_1 M_1 \Lambda_1^{-1} \leq \rho_1 = \|u\|. \end{aligned}$$

Which yields

$$\|Au\| \leq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_{\rho_1}. \tag{3.3}$$

Hence, since $\rho_\star < \rho_1 < \rho^\star$ and from (3.1), (3.2), (3.3), it follows from Theorem 1.2 that A has a fixed point u_1 in $K \cap (\overline{\Omega}_{\rho_1} \backslash \Omega_{\rho_\star})$ and a fixed point u_2 in $K \cap (\overline{\Omega}_{\rho^\star} \backslash \Omega_{\rho_1})$. Both are positive solutions of the BVP (1.1),(1.2) and $0 < \|u_1\| < \rho_1 < \|u_2\|$. The proof is therefore complete. □

Theorem 3.2. *Assume that the following assumptions are satisfied.*

(H3) $f_0 = f_\infty = 0$.

(H4) *There exist constants* $\rho_2 > 0$ *and* $M_2 \in [\Lambda_2, \infty)$ *such that* $f(u) \geq M_2\rho_2$, *for* $u \in [\gamma\rho_2, \rho_2]$.

Then, the problem (1.1)-(1.2) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < \rho_2 < \|u_2\|.$$

Proof. Firstly, since $f_0 = 0$, for any $\epsilon \in (0, \Lambda_1]$, there exists $\rho_\star \in (0, \rho_2)$ such that $f(u) \leq \epsilon u$, for $u \in (0, \rho_\star]$. Let $\Omega_{\rho_\star} = \{u \in E : \|u\| < \rho_\star\}$, then, for any $u \in K \cap \partial\Omega_{\rho_\star}$, we obtain

$$\begin{aligned} Au(t) &\leq \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T T(T-s)a(s)f(u(s))ds \\ &\leq \rho_\star \epsilon \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T T(T-s)a(s)ds \\ &= \rho_\star \epsilon \Lambda_1^{-1} \leq \rho_\star = \|u\|, \end{aligned}$$

which implies

$$\|Au\| \leq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_{\rho_\star}. \tag{3.4}$$

Secondly, in view of $f_\infty = 0$, for any $\epsilon_1 \in (0, \Lambda_1]$, there exists $\rho_0 > \rho_2$ such that

$$f(u) \leq \epsilon_1 u, \quad \text{for } u \in [\rho_0, \infty)\,. \tag{3.5}$$

We consider two cases:

Case (i). Suppose that $f(u)$ is unbounded. Then from $f \in C([0,\infty), [0,\infty))$, we know that there is $\rho^\star > \rho_0$ such that

$$f(u) \leq f(\rho^\star), \quad \text{for } u \in [0, \rho^\star]\,. \tag{3.6}$$

Since $\rho^\star > \rho_0$, then from (3.5), (3.6), one has

$$f(u) \leq f(\rho^\star) \leq \epsilon_1 \rho^\star, \quad \text{for } u \in [0, \rho^\star]\,. \tag{3.7}$$

For $u \in K$ and $\|u\| = \rho^\star$, from (3.7), we obtain

$$\begin{aligned} Au(t) &\leq \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T T(T-s)a(s)f(u(s))ds \\ &\leq \rho^\star \epsilon_1 \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T T(T-s)a(s)ds \\ &= \rho^\star \epsilon_1 \Lambda_1^{-1} \leq \rho^\star = \|u\|, \end{aligned}$$

Case (ii). Suppose that $f(u)$ is bounded, say $f(u) \leq L$ for all $u \in [0,\infty)$. Taking $\rho^\star \geq \max\left\{\frac{L}{\epsilon_1}, \rho_0\right\}$. For $u \in K$ with $\|u\| = \rho^\star$, we have

$$\begin{aligned} Au(t) &\leq \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T T(T-s)a(s)f(u(s))ds \\ &\leq L \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T T(T-s)a(s)ds \\ &\leq \rho^\star \epsilon_1 \frac{2(\beta+1) + T^{-1}\beta\eta(\alpha\eta+2) + \alpha\beta T}{(2T - \alpha\eta^2) - \beta(\alpha\eta^2 - 2\eta + 2T)} \int_0^T T(T-s)a(s)ds \\ &= \rho^\star \epsilon_1 \Lambda_1^{-1} \leq \rho^\star = \|u\|. \end{aligned}$$

Hence, in either case, we always may set $\Omega_{\rho^\star} = \{u \in E : \|u\| < \rho^\star\}$ such that

$$\|Au\| \leq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_{\rho^\star}. \tag{3.8}$$

Finally, set $\Omega_{\rho_2} = \{u \in E : \|u\| < \rho_2\}$. By (H4), for any $u \in K \cap \partial\Omega_{\rho_2}$, we can get

$$\begin{aligned}
Au(\eta) &\geq \frac{2\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)} \int_\eta^T (T-s)a(s)f(u(s))ds \\
&\geq \frac{2\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)} \int_\eta^T (T-s)a(s)M_2\rho_2 ds \\
&\geq \rho_2 \frac{2\eta M_2\gamma}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)} \int_\eta^T (T-s)a(s)ds \\
&= \rho_2 M_2 \Lambda_2^{-1} \\
&\geq \rho_2 = \|u\|,
\end{aligned}$$

which implies

$$\|Au\| \geq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_{\rho_2}. \tag{3.9}$$

Hence, since $\rho_\star < \rho_2 < \rho^\star$ and from (3.4), (3.8) and (3.9), it follows from Theorem 1.2 that A has a fixed point u_1 in $K \cap (\overline{\Omega}_{\rho_2} \backslash \Omega_{\rho_\star})$ and a fixed point u_2 in $K \cap (\overline{\Omega}_{\rho^\star} \backslash \Omega_{\rho_2})$. Both are positive solutions of the BVP (1.1),(1.2) and $0 < \|u_1\| < \rho_2 < \|u_2\|$. The proof is therefore complete. □

4. The existence results of the BVP (1.1),(1.2) for the case: $f_0, f_\infty \notin \{0, \infty\}$

In this section, we discuss the existence for the positive solution of the BVP (1.1),(1.2) assuming $f_0, f_\infty \notin \{0, \infty\}$.

Now, we shall state and prove the following main result.

Theorem 4.1. *Suppose* (H2) *and* (H4) *hold and that* $\rho_1 \neq \rho_2$. *Then, the BVP* (1.1),(1.2) *has at least one positive solution* u *satisfying* $\rho_1 < \|u\| < \rho_2$ *or* $\rho_2 < \|u\| < \rho_1$.

Proof. Without loss of generality, we may assume that $\rho_1 < \rho_2$.

Let $\Omega_{\rho_1} = \{u \in E : \|u\| < \rho_1\}$. By (H2), for any $u \in K \cap \partial\Omega_{\rho_1}$, we obtain

$$\begin{aligned}
Au(t) &\leq \frac{2(\beta+1)+T^{-1}\beta\eta(\alpha\eta+2)+\alpha\beta T}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)} \int_0^T T(T-s)a(s)f(u(s))ds \\
&\leq M_1\rho_1 \frac{2(\beta+1)+T^{-1}\beta\eta(\alpha\eta+2)+\alpha\beta T}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)} \int_0^T T(T-s)a(s)ds \\
&= \rho_1 M_1 \Lambda_1^{-1} \leq \rho_1 = \|u\|,
\end{aligned}$$

which yields

$$\|Au\| \leq \|u\|, \; u \in K \cap \partial\Omega_{\rho_1}. \tag{4.1}$$

Now, set $\Omega_{\rho_2} = \{u \in E : \|u\| < \rho_2\}$. By (H4), for any $u \in K \cap \partial\Omega_{\rho_2}$, we can get

$$\begin{aligned} Au(\eta) &\geq \frac{2\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_\eta^T (T-s)a(s)f(u(s))ds \\ &\geq \frac{2\eta}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_\eta^T (T-s)a(s)M_2\rho_2 ds \\ &\geq \rho_2\frac{2\eta M_2\gamma}{(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T)}\int_\eta^T (T-s)a(s)ds \\ &= \rho_2 M_2 \Lambda_2^{-1} \\ &\geq \rho_2 = \|u\|, \end{aligned}$$

which implies

$$\|Au\| \geq \|u\|, \text{ for } u \in K \cap \partial\Omega_{\rho_2}. \tag{4.2}$$

Hence, since $\rho_1 < \rho_2$ and from (4.1) and (4.2), it follows from Theorem 1.2 that A has a fixed point u in $K \cap (\overline{\Omega}_{\rho_2} \backslash \Omega_{\rho_1})$. Moreover, it is a positive solution of the BVP (1.1),(1.2) and

$$\rho_1 < \|u\| < \rho_2.$$

The proof is therefore complete. □

Corollary 4.2. *Assume that the following assumptions hold.*

(H5) $f_0 = \alpha_1 \in [0, \theta_1\Lambda_1)$, *where* $\theta_1 \in (0, 1]$.

(H6) $f_\infty = \beta_1 \in \left(\frac{\theta_2}{\gamma}\Lambda_2, \infty\right)$, *where* $\theta_2 \geq 1$.

Then, the BVP (1.1),(1.2) *has at least one positive solution.*

Proof. In view of $f_0 = \alpha_1 \in [0, \theta_1\Lambda_1)$, for $\epsilon = \theta_1\Lambda_1 - \alpha_1 > 0$, there exists a sufficiently large $\rho_1 > 0$ such that

$$f(u) \leq (\alpha_1 + \epsilon)u = \theta_1\Lambda_1 u \leq \theta_1\Lambda_1\rho_1, \text{ for } u \in (0, \rho_1].$$

Since $\theta_1 \in (0, 1]$, then $\theta_1\Lambda_1 \in (0, \Lambda_1]$. By the inequality above, (H2) is satisfied. Since $f_\infty = \beta_1 \in \left(\frac{\theta_2}{\gamma}\Lambda_2, \infty\right)$, for $\epsilon = \beta_1 - \frac{\theta_2}{\gamma}\Lambda_2 > 0$, there exists a sufficiently large $\rho_2 (> \rho_1)$ such that

$$\frac{f(u)}{u} \geq \beta_1 - \epsilon = \frac{\theta_2}{\gamma}\Lambda_2, \text{ for } u \in [\gamma\rho_2, \infty),$$

thus, when $u \in [\gamma\rho_2, \rho_2]$, one has

$$f(u) \geq \frac{\theta_2}{\gamma}\Lambda_2 u \geq \theta_2\Lambda_2\rho_2.$$

Since $\theta_2 \geq 1$, $\theta_2\Lambda_2 \in [\Lambda_2, \infty)$, then from the above inequality, condition (H4) is satisfied. Hence, from Theorem 4.1 , the desired result holds. □

Corollary 4.3. *Assume that the following assumptions hold.*

(H7) $f_0 = \alpha_2 \in \left(\frac{\theta_2}{\gamma}\Lambda_2, \infty\right)$, *where* $\theta_2 \geq 1$.

(H8) $f_\infty = \beta_2 \in [0, \theta_1\Lambda_1)$, *where* $\theta_1 \in (0, 1]$.

Then, the BVP (1.1),(1.2) *has at least one positive solution.*

Proof. Since $f_0 = \alpha_2 \in \left(\frac{\theta_2}{\gamma}\Lambda_2, \infty\right)$, for $\epsilon = \alpha_2 - \frac{\theta_2}{\gamma}\Lambda_2 > 0$, there exists a sufficiently small $\rho_2 > 0$ such that

$$\frac{f(u)}{u} \geq \alpha_2 - \epsilon = \frac{\theta_2}{\gamma}\Lambda_2, \text{ for } u \in (0, \rho_2]\,.$$

Thus, when $u \in [\gamma\rho_2, \rho_2]$, one has

$$f(u) \geq \frac{\theta_2}{\gamma}\Lambda_2 u \geq \theta_2\Lambda_2\rho_2.$$

which yields the condition (H4) of Theorem 3.2.

In view of $f_\infty = \beta_2 \in [0, \theta_1\Lambda_1)$, for $\epsilon = \theta_1\Lambda_1 - \beta_2 > 0$, there exists a sufficiently large $\rho_0 (> \rho_2)$ such that

$$\frac{f(u)}{u} \leq \beta_2 + \epsilon = \theta_1\Lambda_1, \text{ for } u \in [\rho_0, \infty)\,. \tag{4.3}$$

We consider the following two cases:

Case (i). Suppose that $f(u)$ is unbounded. Then from $f \in C([0,\infty),[0,\infty))$, we know that there is $\rho_1 > \rho_0$ such that

$$f(u) \leq f(\rho_1), \quad \text{for } u \in [0, \rho_1]\,. \tag{4.4}$$

Since $\rho_1 > \rho_0$, then from (4.3), (4.4), one has

$$f(u) \leq f(\rho_1) \leq \theta_1\Lambda_1\rho_1, \quad \text{for } u \in [0, \rho_1]\,.$$

Since $\theta_1 \in (0,1]$, then $\theta_1\Lambda_1 \in (0, \Lambda_1]$. By the inequality above, (H2) is satisfied.

Case (ii). Suppose that $f(u)$ is bounded, say

$$f(u) \leq L, \quad \text{for all} \quad u \in [0, \infty) \tag{4.5}$$

In this case, taking sufficiently large $\rho_1 > \frac{L}{\theta_1\Lambda_1}$, then from (4.5), we know

$$f(u) \leq L \leq \theta_1\Lambda_1\rho_1, \quad \text{for } u \in [0, \rho_1]\,.$$

Since $\theta_1 \in (0,1]$, then $\theta_1\Lambda_1 \in (0, \Lambda_1]$. By the inequality above, (H2) is satisfied. Hence, from Theorem 4.1, we get the conclusion of Corollary 4.3. □

Corollary 4.4. *Assume that the previous hypotheses* (H2)*,* (H6) *and* (H7) *hold. Then, the BVP* (1.1),(1.2) *has at least two positive solutions u_1 and u_2 such that*

$$0 < \|u_1\| < \rho_1 < \|u_2\|.$$

Proof. From (H6) and the proof of Corollary 4.2, we know that there exists a sufficiently large $\rho_2 > \rho_1$, such that

$$f(u) \geq \theta_2\Lambda_2\rho_2 = M_2\rho_2, \quad \text{for } u \in [\gamma\rho_2, \rho_2]\,,$$

where $M_2 = \theta_2\Lambda_2 \in [\Lambda_2, \infty)$.

In view of (H7) and the proof of Corollary 4.3, we see that there exists a sufficiently small $\rho_2^\star \in (0, \rho_1)$ such that

$$f(u) \geq \theta_2\Lambda_2\rho_2^\star = M_2\rho_2^\star, \quad \text{for } u \in [\gamma\rho_2^\star, \rho_2^\star]\,,$$

where $M_2 = \theta_2\Lambda_2 \in [\Lambda_2, \infty)$.

Using this and (H2), we know by Theorem 4.1 that the BVP (1.1),(1.2) has two positive solutions u_1 and u_2 such that

$$\rho_2^\star < \|u_1\| < \rho_1 < \|u_2\| < \rho_2.$$

Thus, the proof is complete. □

Corollary 4.5. *Assume that the previous hypotheses* (H4)*,* (H5) *and* (H8) *hold. Then, the BVP* (1.1)*,*(1.2) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < \rho_2 < \|u_2\|.$$

Proof. By (H5) and the proof of Corollary 4.2, we obtain that there exists sufficiently small $\rho_1 \in (0, \rho_2)$ such that

$$f(u) \leq \theta_1 \Lambda_1 \rho_1 = M_1 \rho_1, \quad \text{for } u \in (0, \rho_1],$$

where $M_1 = \theta_1 \Lambda_1 \in (0, \Lambda_1]$.

In view of (H8) and the proof of Corollary 4.3, there exists a sufficiently large $\rho_1^\star > \rho_2$ such that

$$f(u) \leq \theta_1 \Lambda_1 \rho_1^\star = M_1 \rho_1^\star, \quad \text{for } u \in [0, \rho_1^\star],$$

where $M_1 = \theta_1 \Lambda_1 \in (0, \Lambda_1]$.

Using this and (H4), we see by Theorem 4.1 that the BVP (1.1),(1.2) has two positive solutions u_1 and u_2 such that

$$\rho_1 < \|u_1\| < \rho_2 < \|u_2\| < \rho_1^\star.$$

This completes the proof. □

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Faouzi Haddouchi, Department of Physics, University of Sciences and Technology of Oran, El Mnaouar, BP 1505, 31000 Oran, Algeria

E-mail address: fhaddouchi@gmail.com

Slimane Benaicha, Department of Mathematics, University of Oran, Es-senia, 31000 Oran, Algeria
E-mail address: slimanebenaicha@yahoo.fr